

# **1.4. Transform-Domain Representation of Discrete Signals and LTI Systems**

## 1.4.1. Z -Transform

**Definition:** The  $Z$  – transform of a discrete-time signal  $x(n)$  is defined as the power series:

$$X(z) = \sum_{k=-\infty}^{\infty} x(n)z^{-k} \qquad X(z) = Z[x(n)]$$

where  $z$  is a complex variable. The above given relations are sometimes called **the direct Z - transform** because they transform the time-domain signal  $x(n)$  into its complex-plane representation  $X(z)$ .

Since  $Z$  – transform is an infinite power series, it exists only for those values of  $z$  for which this series converges. The **region of convergence** of  $X(z)$  is the set of all values of  $z$  for which  $X(z)$  attains a finite value.

The procedure for transforming from  $z$  – domain to the time-domain is called **the inverse  $Z$  – transform**. It can be shown that the inverse  $Z$  – transform is given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad x(n) = Z^{-1}[X(z)]$$

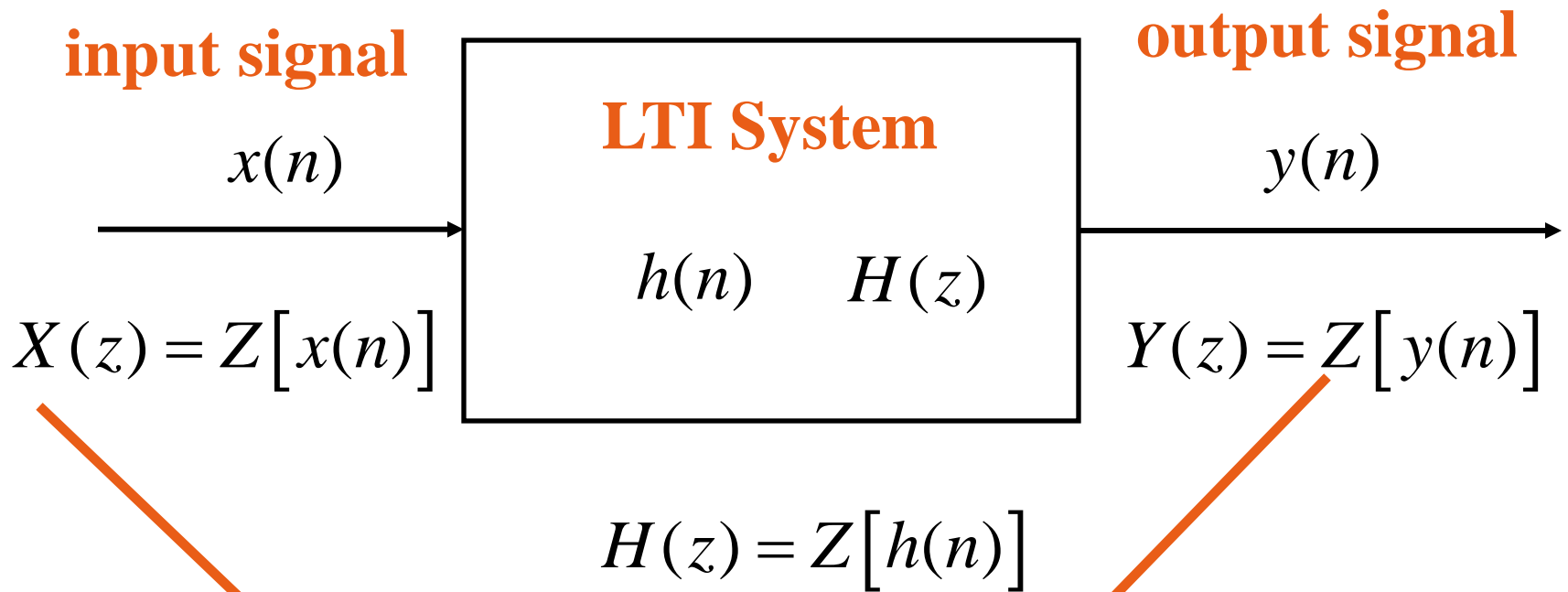
where  $C$  denotes the closed contour in the region of convergence of  $X(z)$  that encircles the origin.

## 1.4.2. Transfer Function

The LTI system can be described by means of **a constant coefficient linear difference equation** as follows

$$y(n) = \sum_{k=0}^N b(k)x(n-k) - \sum_{k=1}^M a(k)y(n-k)$$

Application of the Z-transform to this equation under zero initial conditions leads to the notion of **a transfer function**.



**Transfer function:** the ratio of the  $Z$  - transform of the output signal and the  $Z$  - transform of the input signal of the LTI system:

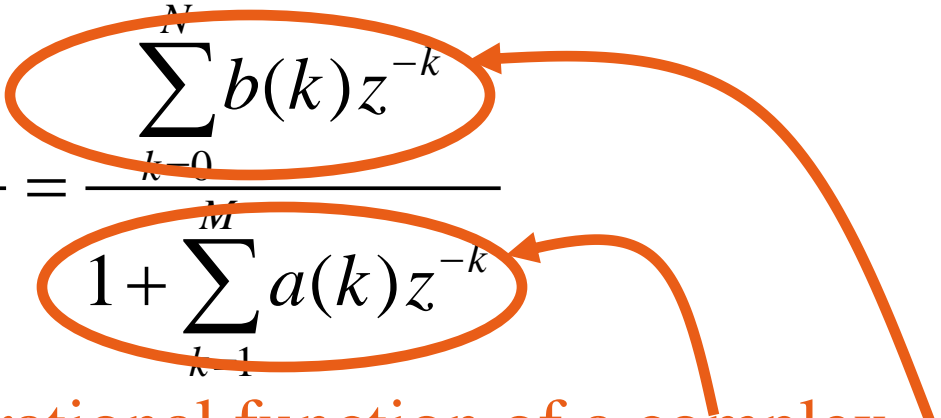
$$H(z) = \frac{Y(z)}{X(z)} = \frac{Z[y(n)]}{Z[x(n)]}$$

LTI system: the Z-transform of the constant coefficient linear difference equation under zero initial conditions:

$$y(n) = \sum_{k=0}^N b(k)x(n-k) - \sum_{k=1}^M a(k)y(n-k)$$

$$Y(z) = \sum_{k=0}^N b(k)z^{-k}X(z) - \sum_{k=1}^M a(k)z^{-k}Y(z)$$

The transfer function of the LTI system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^N b(k)z^{-k}}{1 + \sum_{k=1}^M a(k)z^{-k}}$$


$H(z)$ : may be viewed as a rational function of a complex variable  $z$  ( $z^{-1}$ ).

### 1.4.3. Poles, Zeros, Pole-Zero Plot

Let us assume that  $H(z)$  has been expressed in its irreducible or so-called factorized form:

$$H(z) = \frac{\sum_{k=0}^N b(k)z^{-k}}{1 + \sum_{k=1}^M a(k)z^{-k}} = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^N (z - z_k)}{\prod_{k=1}^M (z - p_k)}$$

**Zeros of  $H(z)$ :** the set  $\{z_k\}$  of  $z$ -plane for which  $H(z_k)=0$

**Poles of  $H(z)$ :** the set  $\{p_k\}$  of  $z$ -plane for which  $H(p_k) \rightarrow \infty$

**Pole-zero plot:** the plot of **the zeros** and **the poles** of  $H(z)$  in the  $z$ -plane represents a strong tool for LTI system description.

**Example:** the 4-th order Butterworth low-pass filter,  
cut off frequency  $\omega_1 = \pi/3$ .

$$b = [ 0.0186 \quad 0.0743 \quad 0.1114 \quad 0.0743 \quad 0.0186 ]$$

$$a = [ 1.0000 \quad -1.5704 \quad 1.2756 \quad -0.4844 \quad 0.0762 ]$$

$$z_1 = -1.0002, z_2 = -1.0000 + 0.0002j$$

$$z_3 = -1.0000 - 0.0002j, z_4 = -0.9998$$

$$H(z) = \frac{\sum_{k=0}^N b(k)z^{-k}}{1 + \sum_{k=1}^M a(k)z^{-k}}$$

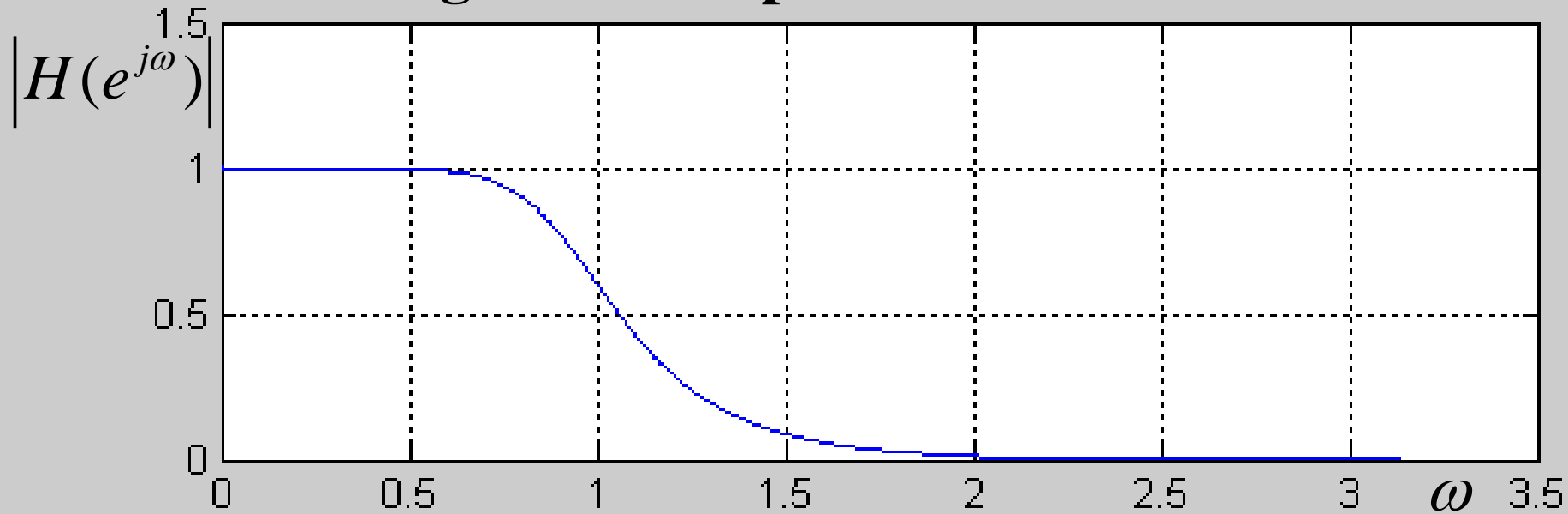
$$H(z) = \frac{\sum_{k=0}^N b(k)z^{-k}}{1 + \sum_{k=1}^M a(k)z^{-k}}$$

$$p_1 = 0.4488 + 0.5707j, p_2 = 0.4488 - 0.5707j$$

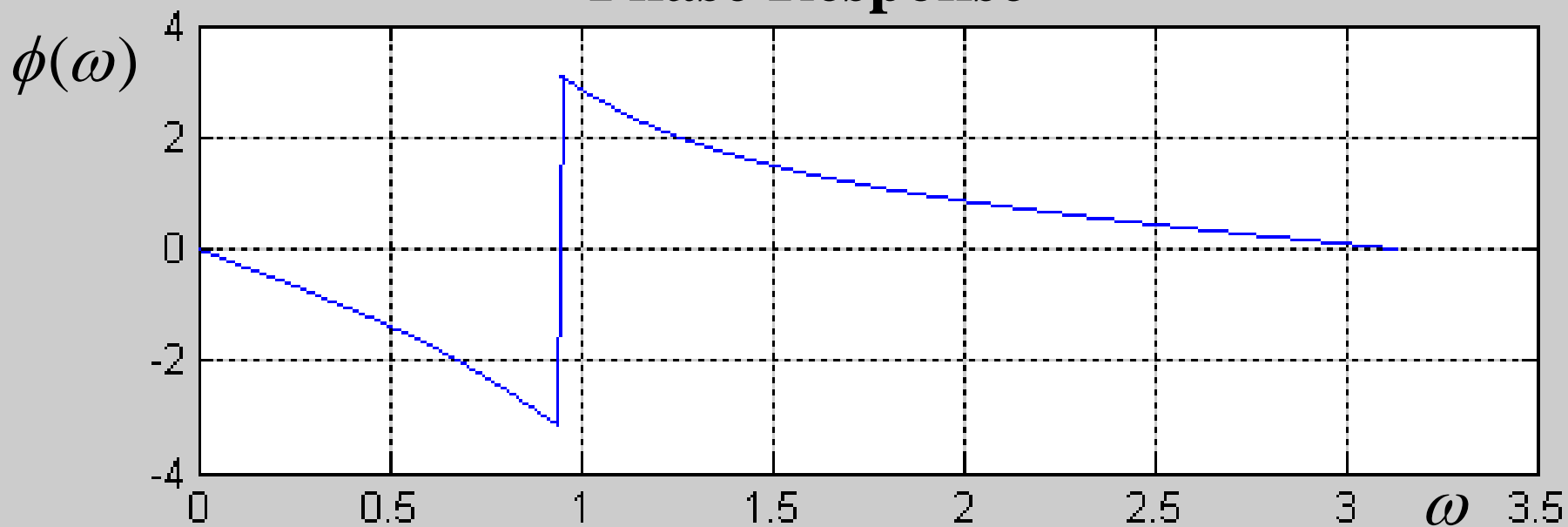
$$p_3 = 0.3364 + 0.1772j, p_4 = 0.3364 - 0.1772j$$



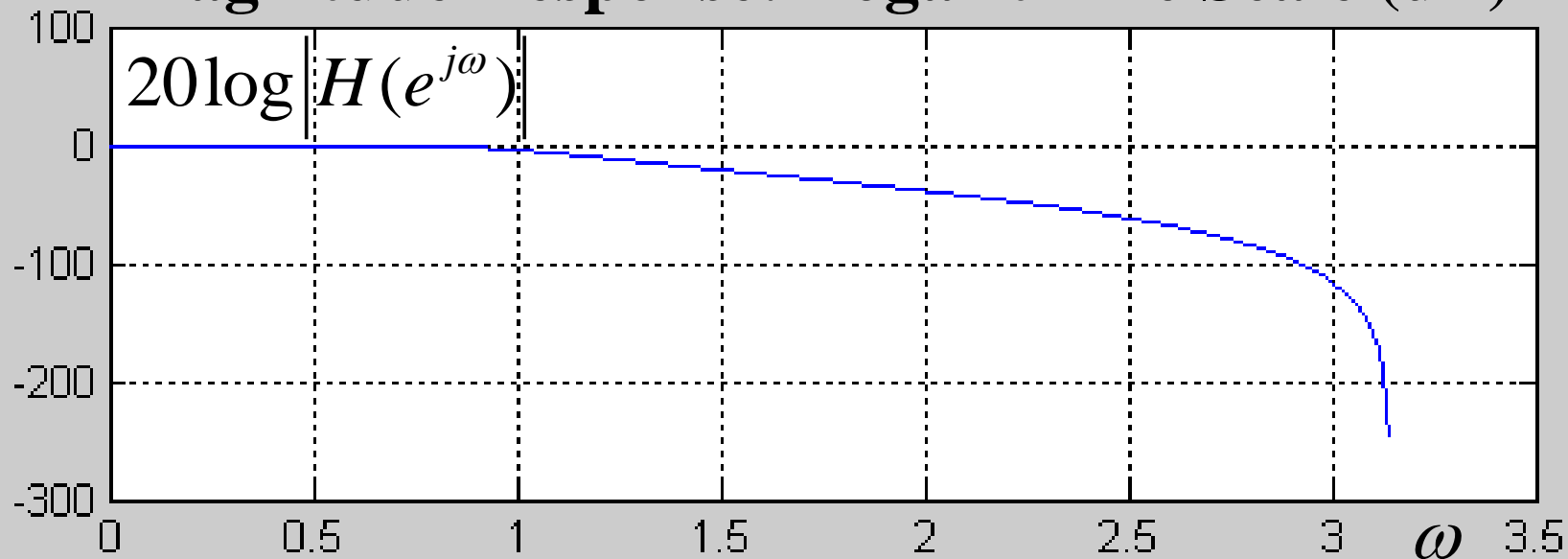
# Magnitude Response: Linear Scale



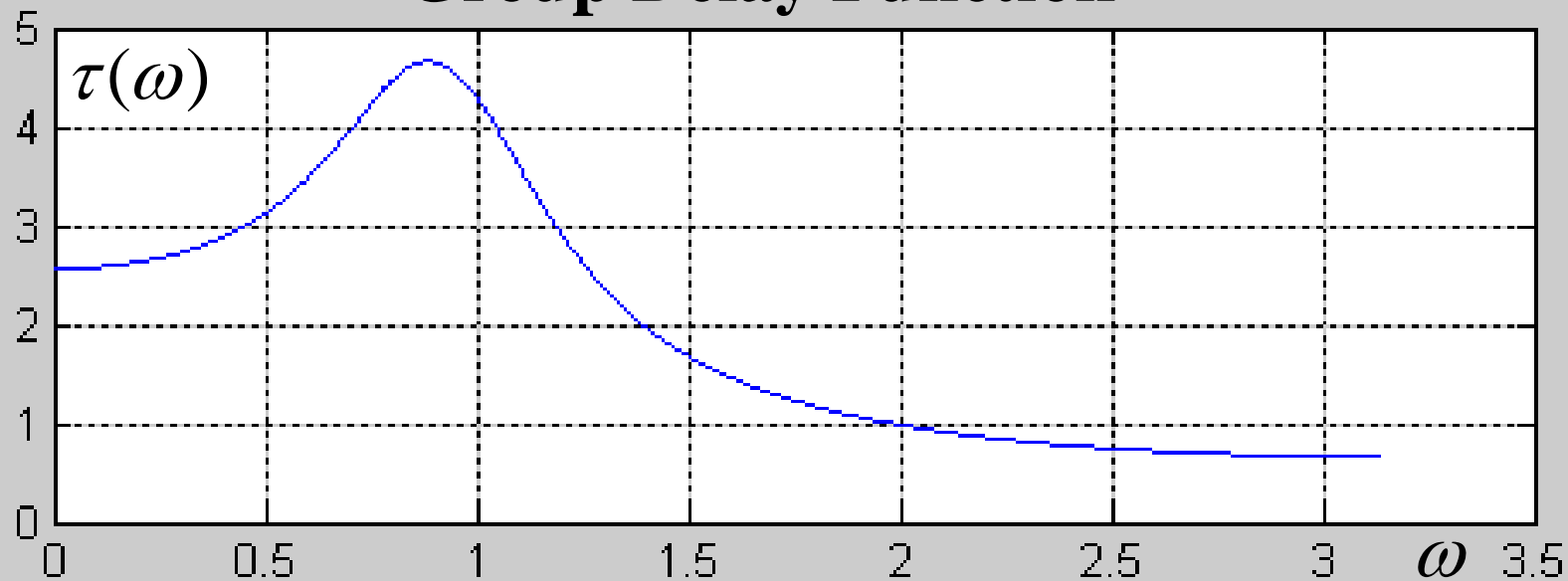
# Phase Response



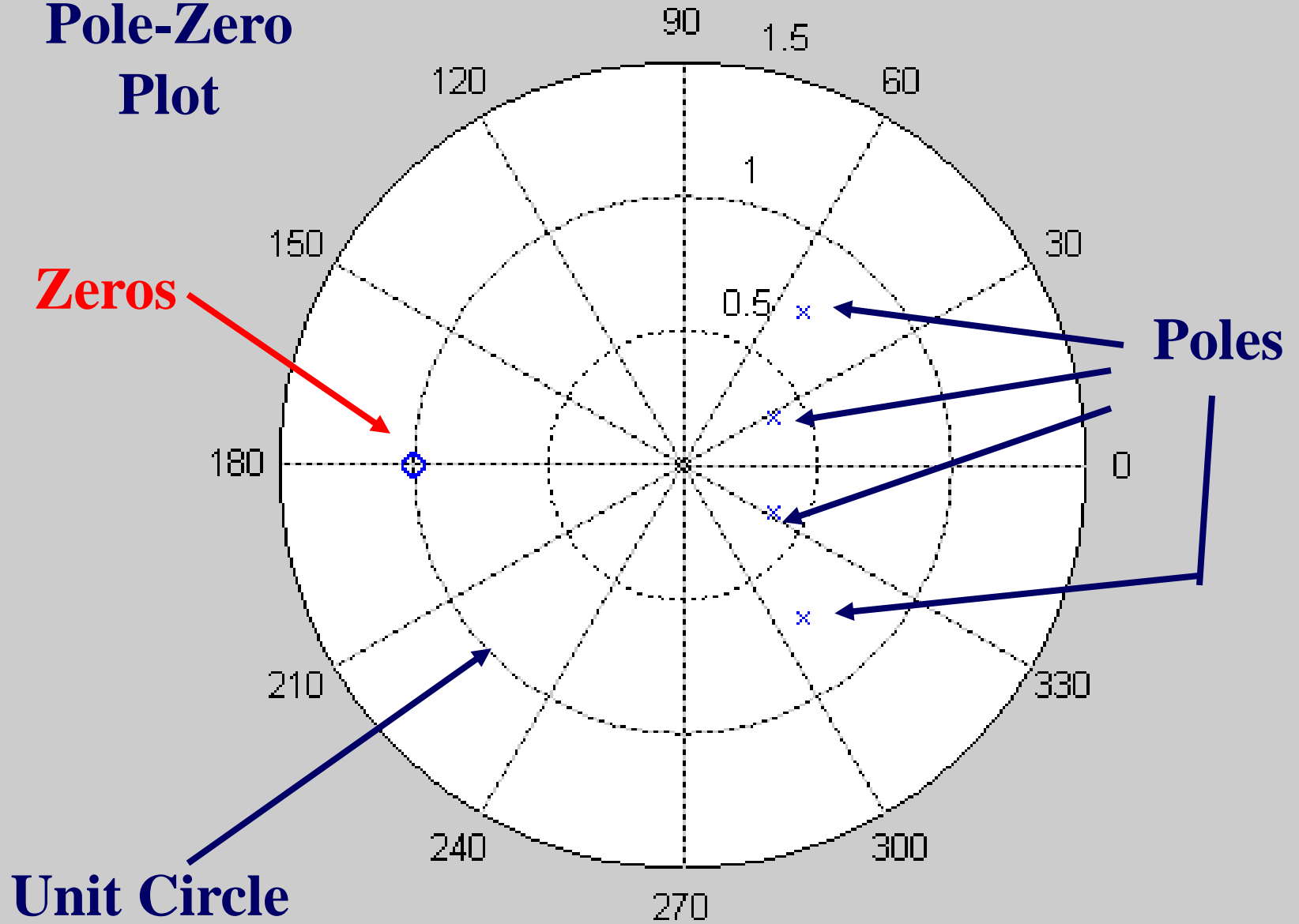
# Magnitude Response: Logarithmic Scale (dB)



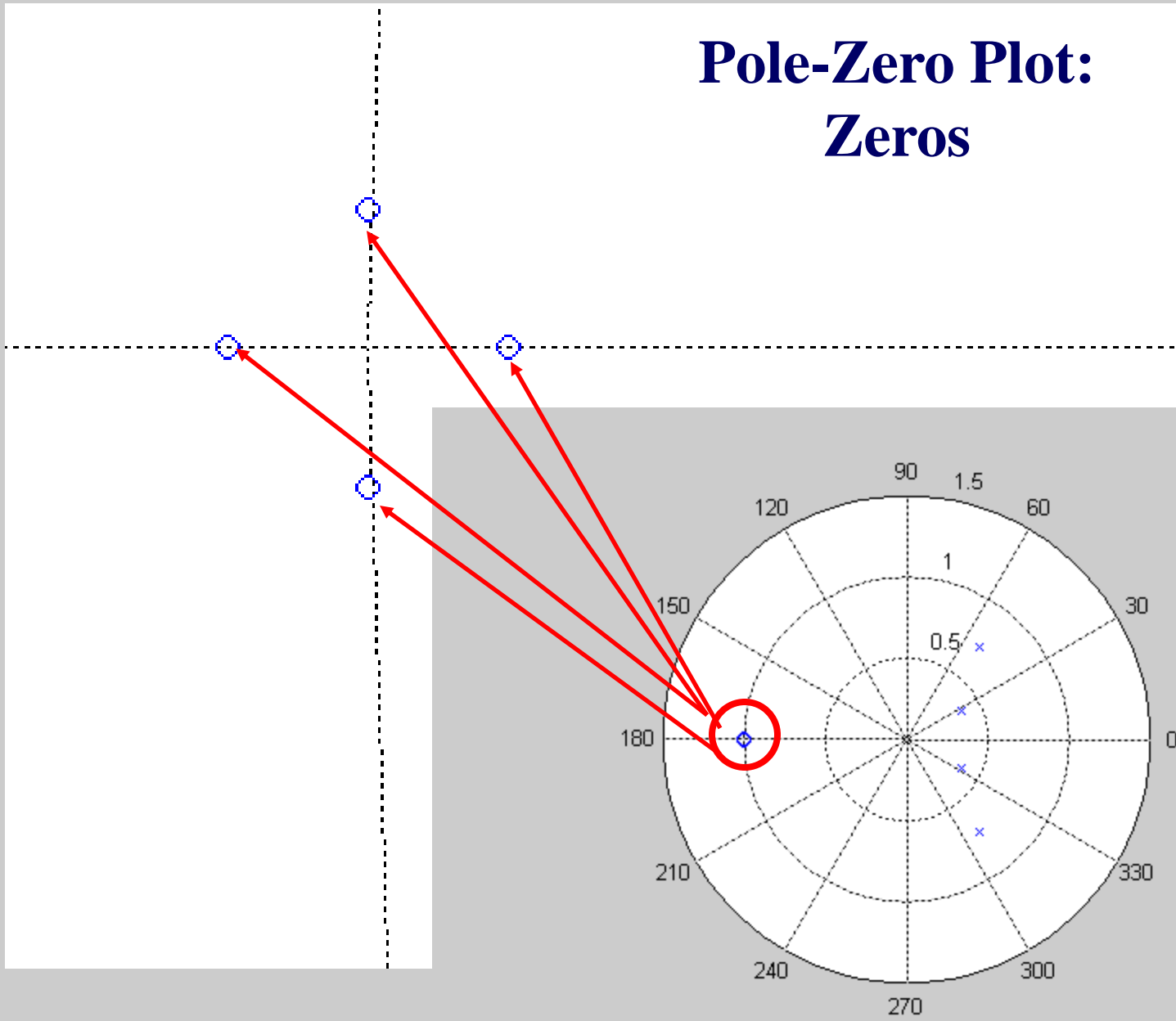
# Group Delay Function



# Pole-Zero Plot



# Pole-Zero Plot: Zeros



## 1.4.4. Transfer Function and Stability of LTI Systems

**Condition:** **LTI system is BIBO stable if and only if** the unit circle falls within the region of convergence of the power series expansion for its transfer function. In the case when the transfer function characterizes a causal LTI system, the stability condition is equivalent to the requirement that **the transfer function  $H(z)$  has all of its poles inside the unit circle.**

## Example 1: stable system

$$H(z) = \frac{1 - 0.9z^{-1} + 0.18z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

$$z_1 = 0.3 \quad p_1 = 0.4000 + 0.6928j \quad |p_1| = 0.8 < 1$$

$$z_2 = 0.6 \quad p_2 = 0.4000 - 0.6928j \quad |p_2| = 0.8 < 1$$

## Example 2: unstable system

$$H(z) = \frac{1 - 0.16z^{-2}}{1 - 1.1z^{-1} + 1.21z^{-2}}$$

$$z_1 = 0.4 \quad p_1 = 0.5500 + 0.9526j \quad |p_1| = 1.1 > 1$$

$$z_2 = -0.4 \quad p_2 = 0.5500 - 0.9526j \quad |p_2| = 1.1 > 1$$

# 1.4.5. LTI System Description. Summary

## Time – Domain:

constant coefficient linear difference equation

$$y(n) = \sum_{k=0}^N b(k)x(n-k) - \sum_{k=1}^M a(k)y(n-k)$$

## Z – Domain:

transfer function

$$H(z) = \frac{\sum_{k=0}^N b(k)z^{-k}}{1 + \sum_{k=1}^M a(k)z^{-k}}$$

## Frequency – Domain:

frequency response

$$H(e^{j\omega}) = \frac{\sum_{k=0}^N b(k)e^{-j\omega k}}{1 + \sum_{k=1}^M a(k)e^{-j\omega k}}$$



**Z**

**FT**

**Z<sup>-1</sup>**

**FT<sup>-1</sup>**

$z = e^{j\omega}$     $e^{j\omega} = z$

**Time – Domain:** impulse response  $h(k)$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \quad H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

**Z – Domain:** transfer function  $H(z)$

$$H(e^{j\omega}) = H(z)_{z=e^{j\omega}} \quad h(n) = \frac{1}{2\pi j} \oint_C H(z) z^{n-1} dz$$

**Frequency – Domain:** frequency response  $H(e^{j\omega})$

$$H(z) = H(e^{j\omega})_{e^{j\omega}=z} \quad h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega$$